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研究ノート

Finite Bubbles in a Non-Bayesian Approach

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＜Abstract＞

This paper presents two players' equilibrium model in which bubbles of security prices occur in finite time even when both players know that the prices are bubbles. We firstly describe a Bayesian model with asymmetric information mainly based on Conlon (2004, Econometrica) and secondly extends it to non-Bayesian setting in which players cannot identify the true probability but a set of probabilities with ambiguity aversion employing epsilon contamination. We proved that in non-Bayesian approach asymmetry of information is not necessary for the existence of bubbles and that bubble prices rise more steeply than those in Bayesian.

＜Keywords＞

Bubble, Bayesian, Conditional probability, Ambiguity, Epsilon contamination

1 Introduction

The bubbles of securities are defined as prices larger than the discounted present values of dividends. Because every player rationally needs to sell these assets before terminal, this phenomenon is difficult to describe as equilibrium models. Many researchers have tackled this problem in a variety of approaches. Among others, Allen, Morris and Postlewaite (1993) describe a situation where all players know that the securities are bubble but each player does not know that other players know that, and proved that bubbles occur even in finite time model in the presence of asymmetric information. Conlon (2004) clarified their model in only two players and succeeded in dropping the assumption that trade pattern must be common knowledge. These researches so far presume that all players know the true probability, which is named Bayesian approach and are based on asymmetric information.

In this paper, we take another approach that all players cannot identify a single probability, which is called non-Bayesian approach or ‘ambiguity’ setting compared to the notion of ‘risk’, which usually refers to the situation where the true probability is known. We employ in this context the epsilon-contamination that is a special case of ambiguity. Several authors have examined the epsilon-contamination or its variants in search behavior (Nishimura and Ozaki, 2004), irreversible investment (Nishimura and Ozaki, 2007) and axiomatization (Nishimura and Ozaki, 2006) to name only a few. In a finite time, two players model with ambiguity we show that bubbles occur without asymmetry of information and that prices of bubbles rise more steeply than those in Bayesian.

Main contributions in this paper are three folds. One is the clarification of the Conlon (2004) in the sense that the probability in our paper is set as a general variable $p\in(0, 1)$ contrary to the specific $p=1/2$ in his paper. Second is that under the ambiguity aversion the equilibrium prices more steeply increase than those in Bayesian, which may be useful to explain sharp rises of security prices in reality. Thirdly, we confirm that if the probability includes the states that they expect to occur but do not realize, a symmetry of information is not necessary for assuring the existence of bubbles even when both players still know that prices are bubbles. This paper treats both the second and the third contributions under the same ambiguity aversion setting.

Logical relation between Bayesian in which the true probability is known and the approach where players have less information is as follows. To assume smaller amount of information or larger one is an alternative approach and either assumption is not stronger nor weaker assumption mathematically. What we concern is the scope of phenomena that we can explain. This paper mainly finds by taking another approach that different prices can be observed as equilibrium bubbles even when both players know that prices are bubbles.

The next section describes a Bayesian model with asymmetric information mainly based on Conlon (2004). The third section provides non-Bayesian model with ambiguity aversion employing epsilon contamination. The last section offers concluding remark.

2　Bayesian Model with Asymmetric Information

Time is assumed to be discrete and written as $t = 0, 1, 2, \cdots T$ where T is finite. Let (Ω, F, P) be a probability space. Denote the stochastic process of the security by $\{S_t\}_{t=0,1,2,\cdots T}: \Omega \to \mathbf{R}_+$ where $\mathbf{R}_+$ stands for the set of non-negative real numbers. Let $q \in(0, 1)$ be a probability of collapse of bubble price of the security to zero at $t = 0,1, \cdots T-1$. Let $p + q = 1$. Because in this paper we only consider the case of bubbles, let us assume that the probability of collapse at $t = T$ equals one. Let $b > 1$, which is more specified later. Define the price of security as follows[1]; $(S_0, S_1, \cdots, S_{T-1}, S_T)=(1, b, b^2, \cdots b^{T-1}, 0)$ with probability p^{T-1}, $(1, b, b^2, \cdots, b^t, 0, \cdots, 0, 0)$ with probability p^tq for $t = 1, 2, \cdots, T-2$, $(1, 0, 0, \cdots, 0)$ with probability q. In this paper x^t or x^i means x to the power of t or i. Same arguments apply to the similar notations. Notice that $q + \sum_{t=1}^{T-2} p^tq + p^{T-1} = 1$ holds.

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[1] For example, if $T = 4$,
$(S_0, S_1, S_2, S_3, S_4)=(1, b, b^2, b^3, 0)$ with probability p^3, $(1, b, b^2, 0, 0)$ with probability p^2q, $(1, b, 0, 0, 0)$ with probability pq, $(1, 0, 0, 0, 0)$ with probability q.

The bubble of security is defined by the security price S_t large than the discounted present value of dividends. In this paper, obeying Conlon (2004), interest rates and dividends are assumed to be zero for simplicity. Thus we can say that the bubble of security occurs when $S_t > 0$ holds for some t. Because S_0=1, the above security prices involve bubbles.

Let us specify the probability space (Ω, F, P) such that for $i = 0, \cdots, T-1$,

$$A_i := \{\omega \in \Omega \mid (S_0, S_1, \cdots, S_{T-1}, S_T) = (1, b, b^2, \cdots, b^i, 0, \cdots, 0, 0)\},$$

$$P(A_0) = q, \qquad P(A_i) = p^i q, \quad i = 1, \cdots T-2, \qquad P(A_{T-1}) = p^{T-1},$$

and

$$F_0 := \{\emptyset, \Omega\}, \qquad F_t := \{A_0, A_1, \cdots, A_{t-1}, \textstyle\bigcup_{i=t}^{T-1} A_i\}, \qquad t = 1, \cdots T-1,$$

$$F := F_{T-1}.$$

Namely, A_i represents the set of states on which the bubble continues until $t=i$. F_t stands for the information available at t so that players can discern whether S_t is b^t or 0 but cannot know S_{t+1} yet.

Consider the following problem;

$$\max \quad E_P[m_T]$$

subject to

$$S_t a_{t+1} + m_{t+1} = S_t a_t + m_t, \qquad t = 0, 1, 2, \cdots, T,$$

$$a_t, m_t \geq 0 \text{ and } a_t, m_t \in \mathbf{N}$$

$$a_0 = 1, \qquad m_0 \geq 1, \quad \text{given,}$$

where a_t stands for the number of securities and m_t represents the quantity of money, which is riskless asset. $a_t, m_t \geq 0$ means that short sales are prohibited following Allen et al (1993) and Conlon (2004), a_t and m_t take only natural number (**N** stands for the set of natural numbers), and each player has one security.

Define

$$b = \frac{1}{p}, \tag{1}$$

namely, the security prices for the sates where bubbles do not collapse are defined by

$$S_t = (1/p)^t$$

for $t = 1, \cdots, T-1$.

Set $\Delta a_{t+1} := a_{t+1} - a_t$ for $t = 0, 1, \cdots T$. Then we can write from $S_T = 0$ and $S_0 = 1$,

$$m_T = S_T(-\Delta a_{T+1}) + S_{T-1}(-\Delta a_T) + S_{T-2}(-\Delta a_{T-1}) \cdots + S_1(-\Delta a_2) + S_0(-\Delta a_1) + m_0$$

$$= S_{T-1}(-\Delta a_T) + S_{T-2}(-\Delta a_{T-1}) \cdots + S_1(-\Delta a_2) + (-\Delta a_1) + m_0.$$

Since a_t is F_{t-1} measurable, we can write

$$a_t(\omega) = a_t^t \quad \text{if } \omega \in \textstyle\bigcup_{i=t-1}^{T-1} A_i,$$

$$a_t(\omega) = a_t^{i+1} \quad \text{if } \omega \in A_i \quad \text{for } i = 0, \cdots, t-2.$$

for $t = 2; \cdots T$ while a_1 is deterministic. So together with the fact that S_{t-1} is F_{t-1} measurable, we see

$$E_P[S_{t-1}\Delta a_t] = b^{t-1}\Delta a_t^t \{ p^{T-1} + \textstyle\sum_{i=t-1}^{T-2} p^i q \} + 0\textstyle\sum_{i=0}^{t-2} \Delta a_t^{i+1}\{p^i q\}$$
$$= b^{t-1}\Delta a_t^t p^{t-1} \qquad (\dagger)$$
$$= \Delta a_t^t$$

Then we have

$$E_P[m_T] = -\Delta a_T^T - \Delta a_{T-1}^{T-1}, \cdots, -\Delta a_2^2 - \Delta a_1 + m_0.$$

Since there are two players and they have only one security, one can reach $\Delta a_t^t = -1$ by selling one security at t. On the other hand, $\Delta a_t^t = -2$ obtained by selling two must accompany $\Delta a_s^s = +1$ by buying one for some $s<t$. Likewise, $\Delta a_t^t = -3$ obtained by selling three must accompany $\Delta a_s^s = +2$ by buying two for some $s<t$, and so on. Therefore we find

$$\max E_P[m_T] = 1 + m_0.$$

Asymmetric Information: Assume that the signal $s \in \{1, \cdots, T-2\}$ is sent to player 1, which makes his or her know that the bubble continues at longest until $t=s$ and collapses afterwards $t>s$, namely, the 'conditional' probability player 1 envisions is; $(S_0, S_1, \cdots, S_{T-1}, S_T) = (1, b, b^2, \cdots, b^s, 0, \cdots, 0)$ with probability p^s, $(1, b, b^2, \cdots, b^t, 0, \cdots, 0, 0)$ with probability $p^t q$ for $t = 1, 2, \cdots, s-1$, $(1, 0, 0, \cdots, 0)$ with probability q, while signal $s+1 \in \{2, \cdots, T-1\}$ is sent to player 2, which informs his or her that the bubble proceeds at most until $s+1$. The `conditional' probability of player 2 is defined by replacing s in the case of player 1 with $s+1$.

According to the asymmetric assumption, player 1 knows the timing of collapse of bubbles one period ahead of that player 2 knows.

The methods that the signal s constructs the above conditional probability can be considered in various ways. This paper offers one method because we are only interested in the consequential conditional probability exhibited above. Let B_i, C_i be disjoint measurable sets such that $A_i = B_i \cup C_i$ and $P(B_i) = p^i q(1-p)$, $P(C_i) = p^{i+1}q$. The signal s makes a player know that he or she stays within $A_s \cup [\bigcup_{i=0}^{s-1} B_i] \subset \bigcup_{i=0}^{T-1} A_i$ sharpening the information from $\bigcup_{i=0}^{T-1} A_i$. Thus we can confirm that the conditional probabilities induced from the signal are as follows;

$$P(B_0 \mid A_s \cup [\textstyle\bigcup_{i=0}^{s-1} B_i]) := q(1-p) \;/\; [p^s q + \textstyle\sum_{i=0}^{s-1} p^i q(1-p)] = q$$
$$P(B_i \mid A_s \cup [\textstyle\bigcup_{i=0}^{s-1} B_i]) := p^i q(1-p) \;/\; [p^s q + \textstyle\sum_{i=0}^{s-1} p^i q(1-p)] = p^i q, \quad i = 1, \cdots, s-1,$$
$$P(A_s \mid A_s \cup [\textstyle\bigcup_{i=0}^{s-1} B_i]) := p^s q \;/\; [p^s q + \textstyle\sum_{i=0}^{s-1} p^i q(1-p)] = p^s.$$

Namely, B_i represents the set of states on which the bubble continues until $t=i$ with conditional probability defied above. Hence we see that under player 1's conditional probability; $(S_0, S_1, \cdots, S_{T-1}, S_T)=(1, b, b^2, \cdots, b^s, 0, \cdots, 0)$ with probability p^s, $(1, b, b^2, \cdots, b^t, 0, \cdots, 0, 0)$ with probability p^tq for $t \leq s-1$, $(1, 0, 0, \cdots, 0)$ with probability q, and under player 2's conditional probability; $(S_0, S_1, \cdots, S_{T-1}, S_T)=(1, b, b^2, \cdots, b^{s+1}, 0, \cdots, 0)$ with probability p^{s+1}, $(1, b, b^2, \cdots, b^t, 0, \cdots, 0, 0)$ with probability p^tq for $t \leq s$, $(1, 0, 0, \cdots, 0)$ with probability q.

This setting is a slight extension of Conlon (2004) that considers only the case of $p = q = 1/2$ while in this paper p and q are put arbitrarily.

Denote the conditional expectation under signal s, $E_P[\, m_T \mid A_s \cup [\bigcup_{i=0}^{s-1} B_i\,]\,]$, by simply $E_P[\, m_T \mid s\,]$. Therefore by obeying the preceding calculations, we have

$$E_P[\, m_T \mid s\,] = -\Delta a_s{}^s - \Delta a_{s-1}{}^{s-1}, \quad \cdots \quad, -\Delta a_2{}^2 - \Delta a_1 + m_0.$$

So similarly to the previous discussion, we obtain

$$\max E_P[\, m_T \mid s\,] = 1 + m_0.$$

The logic of $\max E_P[\, m_T \mid s+1\,] = 1 + m_0$ follows in similar way.

Now let us write the formal definition of the equilibrium.

Definition. An equilibrium of this economy is a set of stochastic processes

$$\{a_{t,1}, m_{t,1}, a_{t,2}, m_{t,2}(S_t)\}_{t=0,1,2,\cdots,T}$$

such that

(e-1) given $\{S_t\}$, $\{a_{t,i}, m_{t,i}\}$ solves the player i's problem for i=1,2;

(e-2) the security market clears; $a_{t,1} + a_{t,2} = 2$; in other words, with $a_{0,1}=1$ and $a_{0,2}=1$ it holds that $\Delta a_{t,1} + \Delta a_{t,2} = 0$ for $t \geq 1$ and $-2 \leq \Delta a_{t,i} \leq 2$ for i=1,2;

(e-3) the money market clears; $m_{t,1} + m_{t,2} = m_0$.

Note that it suffices to prove (e-1) and (e-2) only because the summing up each players' budget constrains deduces (e-3) provided that (e-2) holds (Walras's Law).

For brevity, we abbreviate player's notation i in what follows.

We can see that the equilibrium arises within $A_s \cup [\bigcup_{i=0}^{s-1} B_i\,]$ by the following way (recall (†)):

The case of $s = 1$

player 1: $\Delta a_1=-1$ (selling one) doing nothing afterwards.

player 2: $\Delta a_1=1$ (buying one) anticipating $\Delta a_2^2=-2$ (selling two for price $(1/p)$ with probability p) or $\Delta a_2^1=0$ (doing nothing otherwise) but selling ($\Delta a_2^2=-2$) is not executed ex post.

The case of $s = 2$

player 1: $\Delta a_1 = 1$ (buying one), and $\Delta a_2^2=-2$ (selling two for price $(1/p)$ with probability p) or $\Delta a_2^1=0$ (doing nothing otherwise) doing nothing afterwards.

player 2: $\Delta a_1=-1$ (selling one), and $\Delta a_2^2=-2$ (selling two for price $(1/p)$ with probability p) or $\Delta a_2^1=0$ (doing nothing otherwise) anticipating $\Delta a_3^3=-2$ (selling two for price $(1/p)^2$ with probability p^2) or $\Delta a_3^2=0$ (doing nothing otherwise) but selling ($\Delta a_3^3=-2$) is not executed ex post.

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Generally, player 1 sells at most securities at $t = s$ and player 2 buys them anticipating selling them at $t = s+1$ but fails to undertake ex post satisfying $\max E_P\,[m_T\,|s] = 1 + m_0$ and $\max E_P\,[m_T\,|s+1] = 1 + m_0$.

In effect, we can summarize the above discussion as the existence theorem;

Theorem 1. There exists an equilibrium under each signal s with Bayesian setting.

To make the reader more easily understand the outlook of equilibrium, let us provide a numerical example, which is different from Conlon (2004) as follows.

Numerical Example. Let $T = 4$ and $p=1/3$. Then $b=3$. So we see that the price process is: $(S_0, S_1, S_2, S_3, S_4)=(1, 3, 9, 27, 0)$ with probability 1/27, (1, 3, 9, 0, 0) with probability 2/27, (1, 3, 0, 0, 0) with probability 2/9, (1, 0, 0, 0, 0) with probability 2/3.

The player 1 receives signal $s=3$ and player 2 obtains $s=4$. Thus, for player 1, the price process is $(S_0, S_1, S_2, S_3, S_4)=(1, 3, 9, 0, 0)$ with probability 1/9, (1, 3, 0, 0, 0) with probability 2/9, (1, 0, 0, 0, 0) with probability 2/3, and, for player 2, $(S_0, S_1, S_2, S_3, S_4)=(1, 3, 9, 27, 0)$ with probability 1/27 , (1, 3, 9, 0, 0) with probability 2/27, (1, 3, 0, 0, 0) with probability 2/9, (1, 0, 0, 0, 0) with probability 2/3. The form of equilibrium is the following:

Period 0. player 1 sells one security to player 2 at price 1; The gain: player 1; +1 and player 2; −1.

Period 1. player 1 buys two securities from player 2 when price 3 and does nothing when price 0; The gain: player 1; $2\times-3\times(1/9+2/9)+0\times2/3=-2$ and player 2; $2\times3\times(1/27+2/27+2/9)+0\times2/3=+2$.

Period 2. player 1 sells two securities to player 2 when price 9 and does nothing when price 0; The gain: player 1; $2\times9\times1/9+0\times(2/9+2/3)=+2$ and player 2; $2\times-9\times(1/27+2/27)+0\times(2/9+2/3)=-2$.

Period 3. Player 1 knows that the bubble collapse so does nothing while player 2 anticipates selling two securities to player 1 when price 27 and does nothing when price 0; The gain: player 2; $2\times-27\times1/27+0\times(2/27+2/9+2/3)=+2$.

In effect, the expected total gain of player 1 other than m_0 equals $+1-2+2=1$, and that of player 2 is $-1+2-2+2=1$, which are the maximal of players.

3 Non-Bayesian Model with Ambiguity Aversion

In this section, we consider the case that the players don't know the true probability and only have a set of probabilities. Let us modify the true probability as follows; $(S_0, S_1, \cdots, S_{T-1}, S_T, S_{T+1})=(1, b, b^2, \cdots, b^{T-1}, 0, 0)$ with probability p^{T-1}, $(1, b, b^2, \cdots, b^t, 0, \cdots, 0, 0)$ with probability $p^t q$ for $t = 1, 2, \cdots, T-2$, $(1, 0, 0, \cdots, 0)$ with probability q.

Set the probability Q whose domain is extended from the true probability P as follows; $(S_0, S_1, \cdots, S_{T-1}, S_T, S_{T+1})=(1, b, b^2, \cdots b^T, 0)$ with probability p^T, $(1, b, b^2, \cdots, b^t, 0, \cdots, 0, 0)$ with probability $p^t q$ for $t = 1, 2, \cdots, T-1$, $(1, 0, 0, \cdots, 0)$ with probability q. In Q players optimistically expect that bubbles last one period longer than the true phenomena. In effect, Q includes the states that do not realize. Next we perturb this probability Q. Let $\varepsilon > 0$ with $p-\varepsilon > 0$ and $\eta_k \in (-\varepsilon,\varepsilon)$, $k = 1, \cdots, T$. Write $\eta = (\eta_1, \cdots, \eta_T)$. The new probability $Q_{\boldsymbol{\eta}}$ is defined by perturbing Q in the way that $Q_{\boldsymbol{\eta}}(A_i) =\{\prod_{k=1}^{i} (p+\eta_k)\}(q-\eta_{i+1})$, $Q_{\boldsymbol{\eta}}(A_T) =\prod_{k=1}^{T} (p+\eta_k)$, $\sum_{i=0}^{T} Q_{\boldsymbol{\eta}}(A_i) = 1$ and $Q_{\boldsymbol{\eta}}(A_i)\in(0, 1)$ for $i = 0, \cdots, T$. Note that if p is perturbed to $p+\eta_t$, q is changed to $q-\eta_t$. The set of sates A_T does not really occur but players take into consideration in view of the probability Q. Denote the set of all perturbed probabilities that meet the above conditions by $M\varepsilon$. From construction, all players know that prices are bubbles.

Consider the following problem;

$$\max_{m_T} \min_{Q_\eta \in M\varepsilon} E_{Q_\eta}[m_T]$$

subject to

$$S_t a_{t+1} + m_{t+1} = S_t a_t + m_t, \qquad t = 0, 1, 2, \cdots, T+1,$$

$$a_t,\ m_t \geq 0 \text{ and } a_t,\ m_t \in \mathbf{N}$$

$$a_0 = 1, \quad m_0 \geq 1, \quad \text{given.}$$

Define

$$b = \frac{1}{p-\varepsilon} \tag{2}$$

namely, the security prices for the sates where bubbles do not collapse are defined by

$$S_t=\{1/(p-\varepsilon)\}^t$$

for $t=1, \cdots, T-1$.

We can calculate as follows;

$$E\,Q_\eta[S_{t-1}\Delta a_t] = b^{t-1}\Delta a_t^{\,t}\ \{\ \prod_{k=1}^{T}(p+\eta_k) + \sum_{i=t-1}^{T-1}\ [\prod_{k=1}^{i}(p+\eta_k)]\,(q-\eta_{i+1})\ \}$$

$$+\,0\sum_{i=0}^{t-2}\ \Delta a_t^{\,i+1}\{\ [\prod_{k=1}^{i}(p+\eta_k)]\,(q-\eta_{i+1})\ \}$$

$$= b^{t-1}\Delta a_t^{\,t}\ \prod_{k=1}^{t-1}(p+\eta_k), \qquad (\dagger\dagger)$$

with the convention of $\prod_{k=1}^{0}(p+\eta_k)=1$. Then we obtain,

$$E\,Q_\eta[m_T] = -\,b^{T}\Delta a_{T+1}^{\;T+1}\prod_{k=1}^{T}(p+\eta_k) - b^{T-1}\Delta a_T^{\;T}\prod_{k=1}^{T-1}(p+\eta_k)\,,\ \cdots$$

$$\cdots,\ -\,b^2\Delta a_3^{\,3}(p+\eta_2)(p+\eta_1) - b\Delta a_2^{\,2}\,(p+\eta_1) - \Delta a_1 + m_0.$$

The players select a_t under which η is determined so as to minimize $EQ_\eta[m_T]$. Taking into account the dependence of η, the players choose a_t so as to maximize $EQ_\eta[m_T]$. Although η_k needs to be as high as possible (pessimistically anticipating buying high prices) if players select buying, $\Delta a_t > 0$, they can compensate its minus by selling, $\Delta a_t < 0$, in which case η_k must be as low as possible, namely, $\eta_k = -\varepsilon$. However, buying one security needs to precede selling two securities. Consider buying one at $t = 1$ and selling two at $t = 2$, which yields

$$\min_{\eta_1,\eta_2}\,[\ +\,b^2 2(p+\eta_2)(p+\eta_1) - b(p+\eta_1)$$

$$= b(p+\eta_1)\{2b(p+\eta_2) - 1\}\ ]$$

$$= b(p-\varepsilon)\{2b(p-\varepsilon) - 1\} = 1,$$

since $2b(p-\varepsilon)-1 > 0$ (recall $b = 1/(p-\varepsilon)$). The case of selling only one and buying one is inferior to 1 due to $b(p-\varepsilon)-1 = 0$. In the case of timing other than $t = 1,\ 2$, similar arguments apply since $2b^i(p-\varepsilon)^i-1 > 0$ for $i \geq 2$. To see this, consider buying two at $t = 3$ and selling two at $t = 4$, which yields

$$\min_{\eta_1,\eta_2,\eta_3,\eta_4}\,[\ +\,b^4 2(p+\eta_4)(p+\eta_3)\,(p+\eta_2)(p+\eta_1) - 2b^3(p+\eta_3)\,(p+\eta_2)\,(p+\eta_1)$$

$$= 2b^3(p+\eta_3)\,(p+\eta_2)\,(p+\eta_1)\{b(p+\eta_4) - 1\}\ ]$$

$$= 2b^3(p-\varepsilon)^3\{b(p-\varepsilon) - 1\} = 0,$$

since $b(p-\varepsilon) - 1 = 0$. In the case of timing other than $t = 3, 4$, similar arguments apply since $b^i(p-\varepsilon)^i - 1 = 0$ for $i \geq 2$. Therefore we have

$$\max_{m_T}\ \min_{Q_\eta\in M_\varepsilon} E\,Q_\eta[m_T] = 1 + m_0.$$

The definition of equilibrium in non-Bayesian setting inherits that of the previous section. Then we can see that the equilibrium arises even without asymmetric information in the following way (recall (††));

The case that T is even.

player 1: $\Delta a_1 = 1$ (buying one), and $\Delta a_2^{\,2} = -2$ (selling two for price $(1/(p-\varepsilon))$ with probability $p-\varepsilon$) or $\Delta a_1^{\,2} = 0$ (doing nothing otherwise), and $\Delta a_3^{\,3} = 2$ (buying two for price $(1/(p-\varepsilon))^2$ with probability $(p-\varepsilon)^2$) or $\Delta a_3^{\,2} = 0$ (doing nothing otherwise), and $\Delta a_4^{\,4} = -2$ (selling two for price $(1/(p-\varepsilon))^3$ with probability $(p-\varepsilon)^3$) or $\Delta a_4^{\,3} = 0$ (doing nothing otherwise)

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and $\Delta a_{T\text{-}1}^{T\text{-}1} = 2$ (buying two for price $(1/(p-\varepsilon))^{T\text{-}2}$ with probability $(p-\varepsilon)^{T\text{-}2}$) or $\Delta a_{T\text{-}1}^{T\text{-}2} = 0$ (doing nothing otherwise), and $\Delta a_T^{T}=-2$ (selling two for price $(1/(p-\varepsilon))^{T\text{-}1}$ with probability $(p-\varepsilon)^{T\text{-}1}$) or $\Delta a_T^{T\text{-}1} = 0$ (doing nothing otherwise). That amounts to the repetition of a couple of (buying, selling) totaling T (even number, $0\leq t\leq T\text{-}1$) times of trading.

player 2: player 1's 'buying' is replaced with 'selling', and 'selling' with 'buying' for $t = 0, \cdots, T-1$. In addition, player 2 anticipates at $t = T$, $\Delta a_{T+1}^{T+1} =-2$ (selling two for price $(1/(p-\varepsilon))^{T}$ with probability $(p-\varepsilon)^{T}$) or $\Delta a_{T+1}^{T} = 0$ (doing nothing otherwise) but selling cannot be executed because the occurrence of this state is misunderstood by player 2. That amounts to first selling plus the repetition of a couple of (buying, selling) totaling T+1 (odd number, $0\leq t\leq T$) times of (anticipated) trading.

The case that T is odd.

player 1: $\Delta a_1 = 1$ (buying one), and $\Delta a_2^{2}=-2$ (selling two for price $(1/(p-\varepsilon))$ with probability $p-\varepsilon$) or $\Delta a_1^{2}= 0$ (doing nothing otherwise), and $\Delta a_3^{3}= 2$ (buying two for price $(1/(p-\varepsilon))^2$ with probability $(p-\varepsilon)^2$) or $\Delta a_3^{2}= 0$ (doing nothing otherwise), and $\Delta a_4^{4}=-2$ (selling two for price $(1/(p-\varepsilon))^3$ with probability $(p-\varepsilon)^3$) or $\Delta a_4^{3}= 0$ (doing nothing otherwise)

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and $\Delta a_T^{T} = 2$ (buying two for price $(1/(p-\varepsilon))^{T\text{-}1}$ with probability $(p-\varepsilon)^{T\text{-}1}$) or $\Delta a_{T\text{-}1}^{T\text{-}2} = 0$ (doing nothing otherwise). In addition, player 1 anticipates at $t = T$, $\Delta a_{T+1}^{T+1} =-2$ (selling two for price $(1/(p-\varepsilon))^{T}$ with probability $(p-\varepsilon)^{T}$) or $\Delta a_{T+1}^{T} = 0$ (doing nothing otherwise) but selling cannot be executed because the occurrence of this state is misunderstood by player 1. That amounts to the repetition of a couple of (buying, selling) totaling T+1 (even number, $0\leq t\leq T$) times of (anticipated) trading.

player 2: player 1's 'buying' is replaced with 'selling', and 'selling' with 'buying' for $t = 0, \cdots, T-1$. That amounts to first selling plus the repetition of a couple of (buying, selling) totaling T (odd number, $0\leq t\leq T\text{-}1$) times of trading.

In that process of equilibrium, both players satisfy max min$E_P\ [m_T\ ] = 1+m_0$. Note that the role of player 1 and 2 is interchangeable because both players do not differ in information.

In effect, we can summarize the above discussion as the existence theorem;

Theorem 2. There exists an equilibrium with non-Bayesian setting.

To make the reader more easily understand the outlook of equilibrium under ambiguity aversion setting, let us provide a numerical example as follows.

Numerical Example. Let $T = 4$ (even case), p=1/3 and ε=1/4. Then b=12 (from 1/{1/3−1/4}). Note that b=3 in the Bayesian case. So we see that the price process is: (S_0, S_1, S_2, S_3, S_4, S_5)=(1, 12, 144, 1728, 0, 0) with probability 1/27, (1, 12, 144, 0, 0, 0) with probability 2/27, (1, 12, 0, 0, 0, 0) with probability 2/9, (1, 0, 0, 0, 0, 0) with probability 2/3.

The player 1 and 2 both think the price process as (S_0, S_1, S_2, S_3, S_4, S_5)=(1, 12, 144, 1728, 20736, 0) with probability around 1/81, (1, 12, 144, 1728, 0, 0) with probability around 2/81, (1, 12, 144, 0, 0, 0) with probability around 2/27, (1, 12, 0, 0, 0, 0) with probability around 2/9, (1, 0, 0, 0, 0, 0) with probability around 2/3. Each probabilities are ambiguous, which is connoted in the expression ‘around’. The form of equilibrium is the following:

Period 0. player 2 sells one security to player 1 at price 1; player 1: −1 and player 2: +1.

Period 1. player 2 buys two securities from player 1 when price 12 with probability 1/12 and does nothing when price 0 with some probability; player 1: 2×12×1/12+0×some probability=+2 and player 2: 2×−12×1/12+0×some probability=−2.

Period 2. player 2 sells two securities to player 1 when price 144 with probability 1/144 and does nothing when price 0 with some probability; player 1: 2×−144×1/144+0×some probability=−2 and player 2: 2×144×1/144+0×some probability= +2.

Period 3. player 2 buys two securities from player 1 when price 1728 with probability 1/1728 and does nothing when price 0 with some probability; player 1: 2×1728×1/1728+0×some probability=+2 and player 2: 2×−1728×1/1728+0×some probability=−2 (anticipating selling two securities to player 1 at period 4 when price 20736 with probability 1/20736 and does nothing when price 0 with some probability, so expects to obtain 2×20736×1/20736+0×some probability= +2).

Period 4. Bubble collapses so that player 2 cannot realize his or her expectation.

In effect, the expected total gain of player 1 other than m_0 equals −1+2−2+2=1, and that of player 2 is +1−2+2−2+2=1, which are the maximal of players.

4 Conclusion

This paper presents two players’ equilibrium model in which bubbles of security prices occur in finite time even when both players know that the prices are bubbles. We describe a Bayesian model with asymmetric information and its extension to non-Bayesian setting in which players cannot identify the true probability but a set of probabilities with ambiguity aversion employing epsilon contamination.

We described a Bayesian model with asymmetric information mainly based on Conlon (2004) but more general setting in the sense that the probability of collapsing of bubbles is not specific in our paper while that of Conlon is set to one-half, and proved that in non-Bayesian approach asymmetry of information is not necessary for the existence of bubbles. We also see by comparing (1) and (2) that bubble prices rise more steeply than those in Bayesian.

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